Bounded width and congruence distributivity

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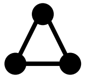
Constraint satisfaction problem (CSP)

Definition. For a finite relational structure $\mathbb{B} = (B; \mathcal{R})$ we define

$$\text{CSP}(\mathbb{B}) = \{\, \mathbb{A} \mid \mathbb{A} \to \mathbb{B} \,\}.$$

- CSP() is the class of three-colorable (directed) graphs.
- CSP() is the class of (directed) bipartite graphs.
- The membership problem for CSP($\mathbb{B}$) is always decidable in nondeterministic polynomial time (**NP**).

Dichotomy Conjecture (Veder, Vardi, 1999)**.** *For every finite structure* $\mathbb{B}$ *the membership problem for* CSP($\mathbb{B}$) *is either in* **P** *or* **NP***-complete.*

Theorem. *The dichotomy conjecture holds if*

- $|B| = 2$ (Schaefer, 1978),
- $\mathbb{B}$ *is an undirected graph* (Hell, Nešetřil, 1990),
- $|B| = 3$ (Bulatov, 2006).

CSP reductions

- If $\mathbb{B} \to \mathbb{C} \to \mathbb{B}$, then $\mathrm{CSP}(\mathbb{B}) = \mathrm{CSP}(\mathbb{C})$.
- We may assume that $\mathbb{B}$ is a **core**, i.e., every endomorphism is an automorphism.
- We may assume that every unary constant relation $\varrho_b = \{b\} \subseteq B$ is in $\mathbb{B}$.
- We may assume that $\mathbb{B}$ is a directed graph with constants.

Definition. $p : B^n \to B$ is a **polymorphism** of $\mathbb{B}$ if every relation of $\mathbb{B}$ is closed under p. Example:

- if $\langle x_1, y_1\rangle, \dots, \langle x_n, y_n\rangle \in R$ then $\langle p(x_1, \dots, x_n), p(y_1, \dots, y_n)\rangle \in R$,
- for ϱ_b this means that $p(b, \dots, b) = b$.

$$\mathrm{Pol}(\mathbb{B}) = \{\, p : B^n \to B \mid p \text{ is a polymorphism of } \mathbb{B}\,\}.$$

- $\mathrm{Pol}(\mathbb{B})$ is a clone; it is idempotent if $\mathbb{B}$ has the unary constant relations.
- If $\mathrm{Pol}(\mathbb{B}) \subseteq \mathrm{Pol}(\mathbb{C})$, then $\mathrm{CSP}(\mathbb{C})$ is polynomial time reducible to $\mathrm{CSP}(\mathbb{B})$.

Nice polymorphisms

Theorem. CSP($\mathbb{B}$) *is in* **P** *if* Pol($\mathbb{B}$) *contains one of the following:*

- *a semilattice operation* (Jevons et. al.)
- *a near-unanimity operation*

$$p(y, x, \ldots, x) \approx p(x, y, x, \ldots, x) \approx \cdots \approx p(x, \ldots, x, y) \approx x,$$

- *a totally symmetric idempotent operation* (Dalmau, Pearson, 1999),
- *a Mal'tsev operation:* $p(x, y, y) \approx p(y, y, x) \approx x$ (Bulatov, 2002; Dalmau, 2004),
- *Generalized majority-minority operation* (Dalmau, 2005),
- *2-semilattices (and conservative algebras)* (Bulatov, 2006),
- *Edge operations* (Idziak, Marković, McKenzie, Valeriote, Willard, 2007),
- *CD(3) Jónsson operations* (Kiss, Valeriote, 2007),
- *CD(4) Jónsson operations* (Carvalho, Dalmau, Marković, Maróti).

Weak near-unanimity

Theorem (Larose, Zádori, 2006). *If* $\mathbb{B}$ *is a core and* $\mathrm{Pol}(\mathbb{B})$ *does not contain a Taylor operation then* $\mathrm{CSP}(\mathbb{B})$ *is* **NP***-complete.*

Theorem (McKenzie, Maróti, 2006). *For a locally finite variety* $\mathcal{V}$ *the TFAE:*

(1) $\mathcal{V}$ *omits type* **1***,*

(2) $\mathcal{V}$ *has a Taylor term,*

(3) $\mathcal{V}$ *has a* **weak near-unanimity** *operation:*

$$p(y, x, \dots, x) \approx \dots \approx p(x, \dots, x, y) \quad \text{and} \quad p(x, \dots, x) \approx x.$$

Corollary. *To prove the dichotomy conjecture it is enough to show that if a core directed graph* $\mathbb{B}$ *has a weak near-unanimity polymorphism then* $\mathrm{CSP}(\mathbb{B})$ *is in* **P***.*

Theorem (Barto, Kozik, Niven, 2007). *The dichotomy conjecture holds for directed graphs without sources and sinks. If* $\mathbb{B}$ *has a weak near-unanimity polymorphism, then the core of* $\mathbb{B}$ *is a disjoint union of circles.*

Local consistency: (j,k)-algorithm

Definition. Let $1 \leq j \leq k$ be integers, and $\mathbb{A}, \mathbb{B}$ be similar relational structures.

Initial step: Put $\mathcal{H}^{(0)} = \bigcup_{\substack{K \subseteq A, \\ |K| \leq k}} \mathcal{H}_K^{(0)}$, where $\mathcal{H}_K^{(0)} = \mathrm{Hom}(\mathbb{A}|_K, \mathbb{B})$.

Iteration step: Let $f : A \to B$ be a partial map, $J \subseteq K$, $|J| \leq j$ and $|K| \leq k$. If one of the following implications does not hold

restriction: $f \in \mathcal{H}_K^{(i)} \Longrightarrow f|_J \in \mathcal{H}_J^{(i)}$,

extension: $f \in \mathcal{H}_J^{(i)} \Longrightarrow \exists g \in \mathcal{H}_K^{(i)},\ g|_{\mathrm{dom}(f)} = f$

then put $\mathcal{H}^{(i+1)} = \mathcal{H}^{(i)} \setminus \{f\}$.

Output: The output of the (j,k)-consistency algorithm is $\mathcal{H}^{(i)}$ if the iteration step cannot be applied.

Definition. A **(j,k)-strategy** is a set $\mathcal{H}$ of partial homomorphisms from $\mathbb{A}$ to $\mathbb{B}$ closed under restrictions and extensions.

- The output $\mathcal{H}$ of the (j,k)-algorithm is always a (j,k)-strategy.

$(1, 2)$-ALGORITHM EXAMPLE

Bounded width

- The (j,k)-algorithm runs in polynomial time (in the size of $\mathbb{A}$).
- The output is independent of the choices made.
- If $\mathcal{H} = \emptyset$, then $\mathbb{A} \not\to \mathbb{B}$.

Definition. The relational structure $\mathbb{B}$ has

(1) **width** (j,k) if $\mathrm{CSP}(\mathbb{B}) = \{\, \mathbb{A} \mid \exists \mathcal{H} \neq \emptyset \ (j,k)\text{-strategy for } \mathbb{A} \text{ and } \mathbb{B} \,\}$,

(2) **total variable width** k if it has width $(k-1,k)$,

(3) "IDB" **width** j if it has width (j,k) for some integer k,

(4) **bounded width** if it has width (j,k) for some j and k.

Lemma. *If $\mathbb{B}$ has bounded width, then* $\mathrm{CSP}(\mathbb{B})$ *is in* **P**, *but not vice verse.*

Theorem (Feder, Vardi, 1998)**.** *TFAE:*

(1) *$\mathbb{B}$ has width (j,k),*

(2) *The complement of* $\mathrm{CSP}(\mathbb{B})$ *is definable in* (j,k)**-Datalog**,

(3) *$\mathbb{B}$ has* (j,k)**-tree duality**.

Bounded width examples

Theorem (Feder, Vardi; Dalmau, Pearson). *A finite relational structure* $\mathbb{B}$ *has width* 1 *if and only if it has a totally symmetric idempotent operation.*

Theorem (Feder, Vardi). *If* $\mathbb{B}$ *has a* $j+1$*-ary near-unanimity polymorphism, then* $\mathbb{B}$ *has width* j.

Example. *The structure* $\mathbb{B} = (\{0,1\}; \varrho, \sigma)$, $\varrho = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle\}$, $\sigma = \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 0,0\rangle\}$ *has width* $(2,3)$ *but does not have width* 1.

- $\mathrm{Pol}(\mathbb{B})$ *is generated by the ternary near-unanimity operation.*
- $\mathrm{Pol}(\mathbb{B})$ *contains no essentially binary operation.*
- $\mathrm{Pol}(\mathbb{B})$ *does not have a totally symmetric operation* p *because otherwise* $q(x,y) = p(x,\dots,x,y)$ *would be a binary commutative operation.*

Theorem (Larose, Zádori). *If* $\mathbb{B}$ *has bounded width, then the variety generated by the algebra* $\mathbf{B} = (B; \mathrm{Pol}(\mathbb{B}))$ *omits types* **1** *and* **2**, *i.e., it is congruence meet-semidistributive.*

Main result

Theorem (Jónsson, 1967)**.** *An algebra* $\mathbf{B}$ *lies in a congruence distributive variety iff there exists an integer* $n > 0$ *and ternary terms* $p_0, \ldots, p_n$ *that satisfy the following identities:*

$$
\begin{aligned}
&p_0(x,y,z) \approx x, && \\
&p_n(x,y,z) \approx z, && \\
&p_i(x,y,x) \approx x && \textit{for all } i, \\
&p_i(x,x,y) \approx p_{i+1}(x,x,y) && \textit{for all even } i, \\
&p_i(x,y,y) \approx p_{i+1}(x,y,y) && \textit{for all odd } i.
\end{aligned}
$$

Theorem. *If* $\mathbb{B}$ *has polymorphisms* $p_0, \ldots, p_4$ *satisfying the above identities then* $\mathbb{B}$ *has width* $(k-1, k)$ *where* k *is the maximum of* 3 *and the largest of the arities of the relations.*

- CD(2) $\Longrightarrow$ majority operation
- CD(3): Kiss and Valeriote proved slightly more: for them k depends only on the size of $\mathbb{B}$, and not on the arities of relations (relational width).

Outline of proof

Put $\mathbf{B} = (B; p_1, p_2, p_3)$. The variety $\mathcal{V} = \mathrm{HSP}(\mathbf{B})$ satisfies the identities:

$$
\begin{aligned}
x &\approx p_1(x,x,y), & p_1(x,y,x) &\approx x,\\
p_1(x,y,y) &\approx p_2(x,y,y), & p_2(x,y,x) &\approx x,\\
p_2(x,x,y) &\approx p_3(x,x,y), & p_3(x,y,x) &\approx x,\\
p_3(x,y,y) &\approx y.
\end{aligned}
$$

- Assume that $\mathbb{B}$ is a directed graph with constants, so $k = 3$.
- Take a nonempty $(2,3)$-strategy $\mathcal{H}$ for $\mathbb{A}$ and $\mathbb{B}$.
- We need to find a map $f : A \to B$ such that $f|_{\{x,y\}} \in \mathcal{H}_{\{x,y\}}$ for all $x, y \in A$.
- If $\mathcal{H}$ is **trivial**, i.e. $|\mathcal{H}_x| = 1$ for all $x \in A$, then $\mathcal{H}$ uniquely determines f.
- If $\mathcal{H}$ is not trivial, then we construct a proper substrategy $\mathcal{H}' \subset \mathcal{H}$.
- In finitely many steps the algorithm must stop (we do not need polynomial time here)

Reduction to ideal free algebras

Definition. Let $\mathbf{C} \leq \mathbf{D} \in \mathcal{V}$.

- $\mathbf{C}$ is a **left-ideal** of $\mathbf{D}$, if $p_2(d, c, c) \in C$ for all $c \in C$ and $d \in D$.
- $\mathbf{C}$ is a **right-ideal** of $\mathbf{D}$, if $p_2(c, c, d) \in C$ for all $c \in C$ and $d \in D$.

Lemma (Kiss, Valeriote)**.** *If $\mathcal{H}$ is a nonempty $(k-1, k)$-strategy, then it has a nonempty $(k-1, k)$-substrategy $\mathcal{H}'$ such that the algebras $\mathcal{H}'_x \in \mathcal{V}$ have no proper left or right-ideals.*

Proof. Assume that $\mathbf{C} < \mathcal{H}_x$ is a proper left-ideal for some $x \in A$.

$$\mathcal{H}' = \{\, f \in \mathcal{H} \mid \forall y, z \in \mathrm{dom}(f)\ \exists f' \in \mathcal{H}_{\{x,y,z\}}\ f'|_{\{y,z\}} = f|_{\{y,z\}},\ f'(x) \in C \,\}.$$

Easy cases: restriction and extension of $f \in \mathcal{H}'_{\{y,z\}}$ to $g \in \mathcal{H}'_{\{x,y,z\}}$.

Interesting case: extension of $f \in \mathcal{H}'_{\{y,z\}}$ to $g \in \mathcal{H}'_{\{y,z,u\}}$.

	x	y	z	u
f	-	b	c	-
f'	a	b	c	-

with $a \in C$ $\overset{?}{\Longrightarrow}$

	x	y	z	u
g	-	b	c	d
g'_1	a_1	-	c	d
g'_2	a_2	b	-	d
g'_3	a_3	b	c	-

with $a_1, a_2, a_3 \in C$.

	x	y	z	u
	-	b	c	d_1
	a	-	c	d_2
	a	b	-	d_3

	x	y	z	u
	-	b	c	d_1
	-	?	c	d_2
	-	b	?	d_3
p_1 :	-	b	c	d

that is $d = p_1(d_1, d_2, d_3)$.

	x	y	z	u
	?	-	c	d_1
	a	-	c	d_2
	a	-	?	d_3
p_1 :	a_1	-	c	d

	x	y	z	u
	?	b	-	d_1
	a	?	-	d_2
	a	b	-	d_3
p_1 :	a_2	b	-	d

$\square$

Reduction to congruence classes

Lemma. *Let $\mathcal{H}$ be a nontrivial $(k-1,k)$-strategy. Then there exists a nonempty set $X \subseteq A$ and maximal congruences $\vartheta_x \in \mathrm{Con}(\mathcal{H}_x)$ for all $x \in X$ such that*

(1) *$\mathcal{H}_{x,y}/(\vartheta_x \times \vartheta_y)$ is the graph of an isomorphism $\tau_{x,y} : \mathcal{H}_x/\vartheta_x \to \mathcal{H}_y/\vartheta_y$ for all $x, y \in X$ of elements,*

(2) *$\tau_{x,y} \circ \tau_{y,z} = \tau_{x,z}$ for all $x, y, z \in X$,*

(3) *$\mathcal{H}_{x,y}/(\vartheta_x \times 0) = (\mathcal{H}_x/\vartheta_x) \times \mathcal{H}_y$ for any $x \in X$ and $y \in A \setminus X$.*

Key step of the proof:

- $x \in X$, $\mathbf{U} = \mathcal{H}_x/\vartheta_x$ simple, has no proper ideal,
- $y \notin X$, $\mathbf{V} = \mathcal{H}_y$ has no proper ideal,
- $\mathbf{R} = \mathcal{H}_{x,y}/(\vartheta_x \times 0)$ is a subdirect product of $\mathbf{U}$ and $\mathbf{V}$,
- $\mathbf{R}$ is not the graph of a homomorphism of $\mathbf{V}$ onto $\mathbf{U}$,

In this case $\mathbf{R} = \mathbf{U} \times \mathbf{V}$.

Entering the right class of ϑ_x

Lemma. *For every $x \in X$ choose a congruence class C_x of ϑ_x such that these correspond to each other via the $\tau_{x,y}$ isomorphism. Let $\mathcal{H}'$ be the set of all functions $f \in \mathcal{H}$ that satisfy the following conditions:*

(1) $f(x) \in C_x$ *for all* $x \in X \cap \operatorname{dom}(f)$,

(2) f *generates a minimal right-ideal in* $\mathcal{H}_{\operatorname{dom}(f)}$.

Then $\mathcal{H}'$ is a $(k-1,k)$-strategy.

Not hard: functions satisfying (2) are always form a strategy.
Key step of the proof:

- $x \in X$, $\mathbf{U} = \mathcal{H}_x/\vartheta_x$ simple, has no proper ideal,
- $y, z \notin X$, $\mathbf{V} = \mathcal{H}_{y,z}$,
- $\mathbf{R} = \mathcal{H}_{x,y,z}/(\vartheta_x \times 0 \times 0)$ is a subdirect product of $\mathbf{U}$ and $\mathbf{V}$,
- $f \in R$, and f generates a minimal right-ideal $\mathbf{S} \leq \mathbf{R}$,

In this case $\mathbf{S} = \mathbf{U} \times \mathbf{S}|_{y,z}$. $\square$

Open problems

- Is it true that every relational structure $\mathbb{B}$ with CD(5) polymorphisms have bounded width?
- Is it true that every relational structure $\mathbb{B}$ with CD(4) polymorphisms must have width $(2, k)$ for some k?
- Is it true that every relational structure $\mathbb{B}$ with a near-unanimity polymorphism (of any arity) must have width $(2, k)$ for some k?
- Is it true that if $\mathbb{B}$ has bounded width then it has width $(2, k)$ for some k?
- Classify subdirect products $\mathbf{R} \leq \mathbf{U} \times \mathbf{V}$ of algebras in a congruence distributive variety where $\mathbf{U}$ is simple and $\mathbf{R}$ is not the graph of a homomorphism of $\mathbf{V}$ onto $\mathbf{U}$.
- What is the smallest directed graph that has a weak near-unanimity polymorphism but does not have bounded width?
- Is there a directed graph that has bounded width but does not have a near-unanimity or totally symmetric idempotent polymorphism?

Bounded width and algebras

Definition. A finite algebra $\mathbf{B}$ has **bounded width** if for every finite set $\mathcal{R} \subset \mathrm{Inv}(\mathbf{B})$ of relations there exist j, k such that $\mathbb{B} = (B; \mathcal{R})$ has width (j, k).

Theorem (Larose, Zádori, 2006). *Every finite algebra in the variety generated by a bounded width algebra has bounded width.*

Definition. A finite algebra $\mathbf{B}$ has **relational width j** if for every finite set $\mathcal{R} \subset \mathrm{Inv}(\mathbf{B})$ of relations $\mathbb{B} = (B; \mathcal{R})$ has width (j, k) where k is the maximum of $j + 1$ and the largest of the arities of the relations.

Definition. A finite algebra $\mathbf{B} = (B; \mathcal{F})$ has **bounded relational width** if it has relational width j for some integer j.

- Is it true that if $\mathbf{B}$ has bounded width then it has bounded relational width?
- Is it true that if $\mathbf{B}, \mathbf{C} \in \mathcal{V}$ have bounded relational width, then so does $\mathbf{B} \times \mathbf{C}$?
- Is it true that if $\mathbb{B}$ has width $(2, k)$ then it has width $(2, k')$ where k' is the maximum of 3 and the largest of the arities of the relations.